Journal of Differential Equations **180**, 171–197 (2002)
doi:10.1006/jdeq.2001.4051, available online at http://www.idealibrary.com on IDEAL®

# Oscillation Theory of Linear Systems

Jacek Tabor

*Institute of Mathematics, Jagiellonian University, Reymonta 4 st., 30-059 Kraków, Poland*
E-mail: tabor@im.uj.edu.pl

Received August 3, 2000

A function $f: \mathbb{R}_+ \to \mathbb{R}$ *strongly oscillates* if for every $t \in \mathbb{R}_+$ such that $f(t) \neq 0$ there exists $s > t$ with $f(t)\, f(s) < 0$. Let $T$ be a strongly continuous semigroup with generator $A$ in a real Banach space $X$. We say that a point $x \in X$ strongly oscillates if for every $\xi \in X^*$ the function $t \to \xi(T(t)\, x)$ strongly oscillates. If the spectrum of the generator $A$ of semigroup $T$ has no nonnegative real values then almost all points in $X$ strongly oscillate. In the case when $A$ generates a strongly continuous group under the above assumption all points strongly oscillate. We apply the above results to strong oscillation of solutions of difference, differential and functional differential equations. 

*Key Words:* oscillation; differential equation; linear system.

## 1. INTRODUCTION

### *Discussion of the Notion of Oscillation*

For reference concerning oscillation of difference and delay difference equations we refer the reader to [5]. According to tradition a given function oscillates if it is neither eventually positive nor eventually negative.

In this paper we will investigate a slightly stronger notion, which we call strong oscillation. The reason to do so is that, as we later show, strong oscillation has a natural geometric interpretation. We need the following definitions.

By $\mathbb{N}$ we denote the set of positive integers. By $\mathbb{T}$ we denote the time parameter, that is $\mathbb{R}$ or $\mathbb{Z}$, while by $\mathbb{T}_+$ we denote $\mathbb{R}_+$ or $\mathbb{Z}_+$.

Definition 1.1. We say that a function $f: \mathbb{T}_+ \to \mathbb{R}$ *strongly oscillates* if for every $t \in \mathbb{T}_+$ such that $f(t) \neq 0$ there exists $s \in \mathbb{T}_+, s > t$ such that $f(s)\, f(t) < 0$.

Now we would like to consider the idea of oscillation of vector valued function. According to [5] a function $f: \mathbb{T}_+ \to \mathbb{R}^n$ oscillates if and only if it oscillates componentwise. However, such defined oscillation clearly



0022-0396/02 $35.00

depends upon the coordinate system. Moreover, it is not well adapted to infinite dimensional Banach spaces. The above reasons led us to the following definition, which in the case of $\mathbb{R}^n$ yields in particular componentwise oscillation.

From now on, if not otherwise specified, $X$ will denote a real Banach space. By $X^*$ we denote the space of all linear functionals on $X$.

DEFINITION 1.2. A function $f: \mathbb{T}_+ \to X$ *strongly oscillates* if for every $\xi \in X^*$ the function $\xi \circ f: \mathbb{T}_+ \to \mathbb{R}$ strongly oscillates.

Let $T: \mathbb{T}_+ \times X \to X$ be a given semidynamical system. For brevity we write $T(t)\, x$ instead of $T(t, x)$. We say that $x \in X$ *strongly oscillates* if the function $\mathbb{T}_+ \ni t \to T(t)\, x \in X$ strongly oscillates.

Now we are ready to describe the contents of our paper. In the next section we obtain general properties of strongly oscillating points. In the third we deal with strong oscillation of solutions of various difference and differential equations with bounded operator coefficients. The last section of our paper is devoted to study of points which strongly oscillate in strongly continuous semigroups. We deal also with strong oscillation of semigroups generated by retarded functional differential equations and by partial differential equations.

*Basic Results from Functional Analysis*

For the convenience of the reader we establish some notation and quote most of the results from the theory of linear operators and strongly continuous semigroups which we apply in the paper. For the following classical definitions and theorems we refer the reader to Appendix II from [2] (see also [11]).

By $X_{\mathbb{C}}$ we denote the complexification of $X$, that is the Cartesian product $X \times X$ with given structure of a complex vector space. For real spaces $X$, $Y$ and operator $A: X \to Y$ by $A_{\mathbb{C}}: X_{\mathbb{C}} \to Y_{\mathbb{C}}$ we denote the complexification of $A$.

Let $Y$ be a Banach space. The space of all bounded linear operators from $Y$ to $Y$ we denote by $\mathscr{L}(Y)$. If $Y$ is a complex Banach space the resolvent set $\rho(A)$ of $A \in \mathscr{L}(Y)$ is defined by

$$\rho(A) := \{z \in \mathbb{C} \mid (zI - A) \text{ is invertible in } \mathscr{L}(X)\},$$

where $I$ denotes the identity operator from $X$ to $X$. The spectrum $\sigma(A)$ is the complement of $\rho(A)$ in $\mathbb{C}$. In the case of real Banach spaces the spectrum of the given operator denotes the spectrum of its complexification.

The spectrum of a bounded operator is always a nonempty compact set. The resolvent $\rho(A) \ni z \to (zI-A)^{-1}$ is a holomorphic function. For every $z \in \mathbb{C}$, $|z| > \|A\|$ we have von Neumann series formula of the resolvent

$$(zI-A)^{-1} = \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} A^n. \tag{1}$$

The exponens of a bounded operator is defined by the formula

$$\exp(A) := \sum_{n=0}^{\infty} \frac{1}{n!} A^n.$$

For every $A \in \mathscr{L}(X)$ we have

$$\sigma(\exp(A)) = \exp(\sigma(A)). \tag{2}$$

A linear operator $A\colon \mathscr{D}(A) \to Y$, with domain $\mathscr{D}(A) \subset Y$, is *closed* if and only if its graph

$$\{(x, Ax) \mid x \in \mathscr{D}(A)\}$$

is a closed subspace of $X \times X$. If $Y$ is a complex Banach space a complex number belongs to the *resolvent set* $\rho(A)$ of an operator $A$ if and only if the *resolvent* $(zI-A)^{-1}$ exists and is bounded. If $Y$ is a real Banach space spectrum of $A$ denotes the spectrum of its complexification.

The spectrum $\sigma(A)$ of a closed operator is by the definition the complement of $\rho(A)$ in $\mathbb{C}$. The resolvent of a closed operator is a holomorphic function on $\rho(A)$. Let $z \in \rho(A)$ be arbitrary. Then $(z+h) \in \rho(A)$ for $h \in \mathbb{C}$ such that $|h| \cdot \|(zI-A)^{-1}\| < 1$ and we have

$$((z+h)\, I-A)^{-1} = \sum_{n=0}^{\infty} h^n(-1)^n\, (zI-A)^{-(n+1)}. \tag{3}$$

Now we are going to quote some definitions and results from the theory of strongly continuous semigroups.

Definition 1.3. We assume that for each $t \in \mathbb{R}_+$ we have $T(t) \in \mathscr{L}(X)$. The family $\{T(t)\}_{t \geqslant 0}$ is a *strongly continuous semigroup* if the following three properties hold:

(i) $T(0) = I$;

(ii) $T(t)\, T(s) = T(t+s)$, for $t, s \in \mathbb{R}_+$,

(iii) for all $x \in X$, $\|T(t)\, x - x\| \to 0$ as $t \to 0$, $t \in \mathbb{R}_+$.

If we exchange in the above definition $\mathbb{R}_+$ with $\mathbb{R}$ then we obtain the definition of the *strongly continuous group*.

The *infinitesimal generator* $A$ of the strongly continuous semigroup $T$ is defined by

$$\mathscr{D}(A)=\left\{x \mid \lim_{h\downarrow 0}\frac{1}{h}(T(h)\,x-x) \text{ exists}\right\},\ Ax=\lim_{h\downarrow 0}\frac{1}{h}(T(h)\,x-x).$$

The infinitesimal generator of a strongly continuous semigroup is a closed operator with dense domain in $X$. If $A$ is a generator of a strongly continuous semigroup then for every $x\in X$ we have

$$\lim_{z\to\infty} z(zI-A)^{-1}(x)=x.$$

There exists the limit

$$type(T):=\lim_{t\to 0}\frac{1}{t}\ln\|T(t)\|$$

which is called the growth bound (or type) of the semigroup $T$. The set $\{z \mid \operatorname{Re} z>type(T)\}$ belongs to $\rho(A)$, and for $z$ in this set, the identity

$$(zI-A)^{-1}x=\int_0^\infty e^{-zs}T(s)\,x\,ds \qquad \text{for} \quad x\in X, \tag{4}$$

holds. We have the following property connecting spectrum of $A$ with spectrum of the semigroup

$$\exp(t\sigma(A))\subset\sigma(T(t)) \qquad \text{for} \quad t\in\mathbb{R}_+. \tag{5}$$

Strongly continuous semigroup $T$ is called uniformly continuous if the function $\mathbb{R}_+\ni t\to T(t)$ is continuous in the operator topology. In this case the generator $A$ of $T$ is a bounded linear operator and $T(t)=e^{tA}$.

## 2. GENERAL RESULTS

### *Geometric Characterization of Strong Oscillation*

In this section we will obtain characterization of strong oscillation of an element in the linear semidynamical system. It will occur that a given point $x$ oscillates if and only if $-x$ belongs to the wedge spanned over the orbit of $x$. This implies in particular that strong oscillation of a given point implies some kind of recurrence.

We first need to explain what we mean by a cone and a wedge. By a *wedge* $W$ in $X$ we mean a closed convex subset of $X$ such that $\alpha W = W$ for every $\alpha > 0$. For a given set $A \subset X$ by $wedge(A)$ we denote the intersection of all wedges containing $A$, that is

$$wedge(A) := cl\left\{\sum \alpha_i a_i \mid \alpha_i \in \mathbb{R}_+,\, a_i \in A\right\}.$$

A wedge $V$ is called a *cone* if $V \cap -V = \{0\}$.

The following geometric characterization of strong oscillation will be often used in the paper. It is important as it connects strong oscillation of the given point with the properties of its orbit. By $orb_T(x)$ we denote the positive orbit (or in other words the positive trajectory) of the point $x$, that is $T(\mathbb{T}_+)\,x$.

THEOREM 2.1. *Let $T\colon \mathbb{T} \times X \to X$ be a linear semidynamical system. Let $x \in X$. Then $x \in X$ strongly oscillates if and only if*

$$-x \in wedge(orb_T(x)). \tag{6}$$

Before we will be ready to present the proof of the above result we first need the following two lemmas.

LEMMA 2.1. *Let $W$ be a wedge in $X$ and let $y \in X \setminus W$. Then there exists $\xi \in X^*$ such that*

$$\xi(W) \subset \mathbb{R}_+$$

*and $\xi(y) < 0$.*

*Proof.* Because $\{y\}$ is a compact convex set by the Banach–Mazur Theorem there exists a functional $\xi \in X^*$ and $\alpha \in \mathbb{R}$ such that $\xi(y) < \alpha$ and

$$\xi(w) > \alpha \qquad \text{for} \quad w \in W.$$

As $0 \in W$ we obtain that $0 > \alpha$, and consequently that $\xi(y) < 0$. Because $W$ is a wedge $kw \in W$ for every $k \in \mathbb{Z}_+$, and so $\xi(w) = \frac{1}{k}\,\xi(kw) > \frac{1}{k}\,\alpha$ for every $k \in \mathbb{Z}_+$, which implies that $\xi(w) \geqslant 0$ for every $W \in W$. ∎

We omit proof of the following easy lemma.

LEMMA 2.2. *Let $X, Y$ be Banach spaces, let $x \in X$ and let $S \subset X$. Let $A\colon X \to Y$ be a bounded operator. Then*

$$A(wedge(S)) \subset wedge(A(S)).$$

*Proof* (of Theorem 2.1). Suppose that (6) holds. We will show that $x$ strongly oscillates. Let $\xi \in X^*$ be arbitrary and let $t \in \mathbb{T}_+$ be such that $\xi(T(t)\,x) > 0$. Let us assume for contradiction that $\xi(x(s)) \in \mathbb{R}_+$ for every $s \geqslant t$. By (6) and Lemma 2.2 we obtain that

$$-T(t)\,x \in T(t)(wedge(orb_T(x))) \subset wedge(orb_T(T(t)\,x)).$$

Applying once more Lemma 2.2 we obtain that

$$\begin{aligned}0 > -\xi(T(t)) &\geqslant \inf\{\xi(wedge(orb_T(T(t)\,x))\}\\ &\geqslant \inf\{wedge\{\xi(orb_T(T(t)\,x))\} \geqslant 0,\end{aligned}$$

a contradiction.

We show the inverse implication. For an indirect proof let us suppose that $x$ strongly oscillates and that (6) does not hold. Then by Lemma 2.1 there exists $\xi \in X^*$ such that $\xi(-x) < 0$ and $\xi(wedge(orb_T(x))) \subset \mathbb{R}_+$. This means that $\xi(x) > 0$ and $\xi(T(s)\,x) \geqslant 0$ for every $s \in \mathbb{T}_+$, which contradicts the assumption that $x$ strongly oscillates. ∎

The following corollary of Theorem 2.1 shows that if a given point strongly oscillates then its trajectory can not be contained in a cone.

COROLLARY 2.1. *Let $T$ be a linear semidynamical system in $X$ and let $V$ be a cone in $X$. Suppose that $x \in V \setminus \{0\}$ strongly oscillates. Then there exists $t \in \mathbb{T}_+$ such that $T(t)\,x \notin V$.*

*Proof.* For an indirect proof let us assume that $orb_T(x) \subset V$. Because $x$ strongly oscillates by Theorem 2.1 $-x \in wedge(orb_T(x)) \subset V$. Thus $x \in V \cap -V = \{0\}$, a contradiction. ∎

*Backward Continuation*

In the following section we are going establish the main tool which we will apply in the investigation of systems with discrete and continuous time.

The main idea is to "continue back" the resolvent of a given operator. The following theorem is a direct generalization of Lemma 1 from [12].

THEOREM 2.2. *Let $\mathscr{V} \subset \mathscr{L}(X)$ be a wedge such that*

$$B, C \in \mathscr{V} \Rightarrow B \circ C \in \mathscr{V}. \tag{7}$$

*Let $A$ be a closed operator such that $\sigma(A) \cap [\alpha, \beta] = \varnothing$ for certain $\alpha, \beta \in \mathbb{R}$ and that*

$$(\beta I - A)^{-1} \in \mathscr{V}.$$

*Then*

$$(\alpha I - A)^{-1} \in \mathscr{V}.$$

*Proof.* Let

$$K := \{k \in [\alpha, \beta] : (kI - A)^{-1} \in V\}.$$

As the resolvent is a continuous function and $\mathscr{V}$ is closed we obtain that $K$ is closed. Let $k_0 := \inf\{K\}$.

For an indirect proof let us suppose that $k_0 > \alpha$. We know that the resolvent of $A$ satisfies the following equation

$$((k_0 - h)\, I - A)^{-1} = \sum_{n=0}^{\infty} h^n (k_0 I - A)^{-(n+1)} \tag{8}$$

for $h \in \mathbb{C}$ such that $|h| \cdot |(k_0 I - A)^{-1}| < 1$. We choose $h \in (0, \infty)$ such that $h < k_0 - \alpha$ and $h \cdot |(k_0 I - L)^{-1}| < 1$. By (7), (8) and the fact that $\mathscr{V}$ is a wedge we obtain that $((k_0 - h)\, I - A)^{-1} \in \mathscr{V}$, and consequently that $k_0 - h \in K$. We have obtained a contradiction with the fact that $k_0 = \inf\{K\}$. ∎

Now we would like to present some corollaries of the above theorem.

COROLLARY 2.2. *Let $A \in \mathscr{L}(X)$ be such that $\sigma(A) \cap \mathbb{R}_+ = \varnothing$. Then*

$$-A^{-1} \in \textit{wedge}\{A^n \colon n \in \mathbb{Z}_+\}.$$

*Proof.* Let $\mathscr{V} := \textit{wedge}\{A^n \colon n \in \mathbb{Z}_+\}$. One can easily check that $\mathscr{V}$ satisfies condition (7). Applying the von Neumann formula we obtain that

$$(2\, \|A\|\, I - A)^{-1} = \sum_{n=0}^{\infty} \frac{1}{(2\, \|A\|)^{n+1}} A^n,$$

which implies that $(2\, \|A\|\, I - A)^{-1} \in \mathscr{V}$. By the assumptions trivially $\sigma(A) \cap [0, 2\, \|A\|] = \varnothing$. Making use of Theorem 2.2 we thus obtain that $-A^{-1} = (0 - A)^{-1} \in \mathscr{V}$. ∎

PROPOSITION 2.1. *Let $T$ be a strongly continuous semigroup in a Banach space $X$, and let $A$ denote its generator. We assume that $\sigma(A) \cap \mathbb{R} = \varnothing$. Then*

$$-(\alpha I + A)^{-1}\, x \in \textit{wedge}\{T(t)\, x \mid t \in \mathbb{R}_+\},$$

*for every $\alpha \in \mathbb{R}$, $x \in X$.*

*Proof.* Let $V := \textit{wedge}\{T(t)\, x \mid t \in \mathbb{R}_+\}$.

Let $\alpha \in \mathbb{R}$ be fixed and let $\omega \in \mathbb{R}_+$, $\omega > \max\{type(T), -\alpha\}$ be chosen arbitrarily. We know that

$$(\omega I - A)^{-1} x = \int_0^\infty e^{-\omega t} T(t)\, x\, dt,$$

which yields that $(\omega I - A)^{-1} x \in V$. Making use of the reasoning similar to that from Theorem 2.2 and the fact that $\sigma(A) \cap [-\alpha, \omega] = \varnothing$ we obtain that $-(\alpha I + A)^{-1} x = (-\alpha I - A)^{-1} x \in V$. ▮

## 3. DIFFERENCE AND DIFFERENTIAL EQUATIONS

*First Order Systems*

Let $A \in \mathscr{L}(X)$ be fixed. Then $A$ generates a semidynamical system with discrete time by the formula $T(n) := A^n$.

THEOREM 3.1. *Let $A \in \mathscr{L}(X)$ be such that*

$$\sigma(A) \cap \mathbb{R}_+ = \varnothing.$$

*Then every point of $X$ strongly oscillates in the discrete semidynamical system generated by $A$.*

*Proof.* Applying Corollary 2.2 we obtain that $-A^{-1} \in wedge\{A^n \mid n \in \mathbb{Z}_+\} =: \mathscr{V}$. As $A \in \mathscr{V}$ we have $(-A)^{-1} \circ A \in \mathscr{V}$, and therefore

$$-I \in wedge\{A^n \mid n \in \mathbb{Z}_+\}. \tag{9}$$

Let $x \in X$ be arbitrarily chosen and let $P_x \colon \mathscr{L}(X) \to X$ be defined by $P_x(B) := B(x)$. Making use of Lemma 2.2 and (9) we get

$$-x \in wedge\{A^n x \mid n \in \mathbb{Z}_+\} = wedge\{orb_A(x)\}.$$

By Theorem 2.1 this means that $x$ strongly oscillates. ▮

The following simple result states partial inverse of the previous theorem (for operators which spectrum coincides with point spectrum).

PROPOSITION 3.1. *Let $A \in \mathscr{L}(X)$ be such that $\sigma_p(A) \cap \mathbb{R}_+ \neq \varnothing$. Then there exists an $x \in X$ which does not strongly oscillate.*

*Proof.* Let $\alpha \in \sigma_p(A) \cap \mathbb{R}_+$ be chosen arbitrarily. Then there exists $x \in X \setminus \{0\}$ such that $A(x) = \alpha x$. As $\alpha \geqslant 0$ we obtain that $x$ does not strongly oscillate. ▮

In the finite dimensional case clearly $\alpha \in \sigma(A)$ if and only if $\alpha \in \sigma_p(A)$. Thus in this case Theorem 3.1 gives us a characterization of discrete semigroups which every point strongly oscillates. As shows the following example in the infinite dimensional case the situation is different—there exist operators for which every point strongly oscillates, but which spectrum is not disjoint with $\mathbb{R}_+$.

EXAMPLE 3.1. By $l^2$ we denote the Hilbert space of real valued sequences with the norm $\|(x_1, ..., x_n, ...)\|^2 = \sum_{i=1}^{\infty} x_i^2$.

We define $A \in \mathscr{L}(l^2)$ by

$$A(x_1, ..., x_n, ...) = \left(-\frac{1}{2^1}\, x_1, ..., -\frac{1}{2^n}\, x_n, ...\right).$$

One can easily notice that

$$\sigma(A) = \{0\} \cup \bigcup_{i=1}^{\infty} \left\{-\frac{1}{2^i}\right\}.$$

This yields in particular that $\sigma(A) \cap \mathbb{R}_+ \neq \varnothing$.

However, we show that every point of $l^2$ strongly oscillates. Let $x = (x_1, ..., x_n, ...) \in l^2$ and $\xi \in (l^2)^*$ be arbitrary. Let

$$m := inf\{n \in \mathbb{Z}_+ \mid \xi(0, ..., 0, \overset{n}{\widehat{x_n}}, 0, ...) \neq 0\}.$$

If $m = +\infty$ then by the continuity of $\xi$ and the definition of $A$ we obtain that $\xi(A^k x) = 0$ for all $k \in \mathbb{Z}_+$, so $\xi(A^k x)$ strongly oscillates.

So let us now consider the remaining case when $m \in \mathbb{Z}_+$. Then

$$\begin{aligned}
&\|(-2^m)^k\, \xi(A^k x) - \xi(0, ..., 0, x_n, 0, ...)\| \\
&\quad \leqslant \|(-2^m)^k\, \xi(A^k(x_1, ..., x_{m-1}, 0, 0, ...)\| \\
&\qquad + \|(-2^m)^k\, \xi(A^k(0, ..., 0, x_{m+1}, x_{m+2}, ...)\| \\
&\quad = \|(2^m)^k\, \xi(A^k(0, ..., 0, x_{m+1}, x_{m+2}, ...)\| \\
&\quad \leqslant \|\xi\| \cdot \left\|(2^m)^k \left(0, ..., 0, \left(\frac{1}{2^{m+1}}\right)^k x_{m+2}, \left(\frac{1}{2^{m+2}}\right)^k x_{m+2}, ...\right)\right\| \\
&\quad = \|\xi\| \cdot \left\|\left(0, ..., 0, \left(\frac{1}{2}\right)^k x_{m+1}, \left(\frac{1}{2^2}\right)^k x_{m+2}, ...\right)\right\| \leqslant \frac{1}{2^k}\, \|\xi\| \cdot \|x\|.
\end{aligned}$$

This implies that

$$\lim_{k\to\infty} (2^m)^{2k}\, \xi(A^{2k}x) = \xi(0, \ldots, 0, x_m, 0, \ldots),$$

$$\lim_{k\to\infty} (2^m)^{2k+1}\, \xi(A^{2k+1}x) = -\xi(0, \ldots, 0, x_m, 0, \ldots).$$

As $\xi(0, \ldots, 0, x_m, 0, \ldots) \neq 0$ this yields that $\xi(A^k x)$ strongly oscillates.

There appears an interesting question whether it is possible to construct an operator $A$, $\sigma(A) \cap (0, \infty) \neq \varnothing$, such that all points strongly oscillate in the semidynamical system generated by $A$.

Now we would like to show that Theorem 3.1 can be applied to investigation of the uniformly continuous semigroups. Later on by different methods we will obtain a generalization of this result.

COROLLARY 3.1. *Let $T$ be a uniformly continuous semigroup and let $A \in \mathscr{L}(X)$ be its generator. If $\sigma(A) \cap \mathbb{R} = \varnothing$ then every point in $X$ strongly oscillates.*

*Proof.* We know that

$$T(t)\, x = \exp(tA)\, x.$$

Since $\sigma(A)$ is compact there exists $t_0 > 0$ such that

$$t_0 \sigma(A) \subset \mathbb{R} \times (-\tfrac{1}{2}\pi, \tfrac{1}{2}\pi). \tag{10}$$

Let $a + ib \in t_0 \sigma(A)$, where $a, b \in \mathbb{R}$, be arbitrary. As $\sigma(A) \cap \mathbb{R} = \varnothing$ we obtain that $b \neq 0$. Moreover, by (10) we get that $b \in (-\frac{1}{2}\pi, \frac{1}{2}\pi)$, and therefore $\sin(b) \neq 0$. Thus

$$\exp(a+ib) = \exp(a)(\cos(b) + i \sin(b)) \in \mathbb{C} \setminus \mathbb{R}.$$

Let $B := \exp(t_0 A)$. Then $\sigma(B) = \sigma(\exp(t_0 A) \subset \mathbb{C} \setminus \mathbb{R}_+$, so by Theorem 3.1 we obtain that every point of $X$ strongly oscillates in the semidynamical system generated by $B$. This yields in particular that every point strongly oscillates in the system $T$. ∎

*Global Equivalence*

To proceed further we need some results from [4] (see also [10]). Let us first introduce the following definition.

DEFINITION 3.1. Let $W\colon \mathbb{C} \to \mathscr{L}(X)$ be a holomorphic function. We define $\varsigma(W)$, the *spectrum* of $W$ by the formula

$$\varsigma(W) := \{z \in \mathbb{C} \mid W(z) \text{ is not invertible in } \mathscr{L}(X)\}.$$

Analogously we define the *point spectrum* of $W$ by

$$\varsigma_p(W) := \{z \in \mathbb{C} \mid \exists x \in X \setminus \{0\} : W(z)\, x = 0\}.$$

If for a given element $A \in \mathscr{L}(X)$ we put $W_A(z) := zI - A$, then $\sigma(A)$ coincides with $\varsigma(W_A)$, and $\sigma_p(A)$ with $\varsigma_p(W_A)$.

Suppose that we are given a linear scalar differential equation of $n$th order $d^n y/dt^n = \sum_{i=0}^{n-1} a_i\, d^i y/dt^i$. Then it is well known that this equation can be "modified" to the system of $n$-linear equations of the first order. The idea is to introduce new variables $z_0, ..., z_{n-1}$ by $z_i(t) = d^i y/dt^i$. The main advantage of this operation is that study of linear equation of $n$th order is reduced to study of the equation of first order (however of higher dimension).

A similar procedure can be adapted to equations with operator coefficients.

For polynomial $W(z) = Iz^n - \sum_{i=0}^{n-1} A_i z^i$, where $A_i \in \mathscr{L}(X)$, we define the operator $\mathscr{P}(W) \in \mathscr{L}(X^n)$ by

$$\mathscr{P}(W) := \begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & I \\ A_0 & A_1 & \cdots & \cdots & A_{n-1} \end{bmatrix}$$

The operator $\mathscr{P}(W)$ is a formalization of the idea of changing the linear equation of $n$th order to a linear equation of first order. Directly from the definition of $\mathscr{P}(W)$ one can check that

$$\sigma_p(\mathscr{P}(W)) = \varsigma_p(W).$$

The following result from [4] is essential to applications of the previous subsection to the difference and differential equations.

*Global Equivalence.* Let $W(z) = Iz^n - \sum_{i=0}^{n-1} A_i z^i$, where $A_i \in \mathscr{L}(X)$, be arbitrary. Then there exist holomorphic functions $E, F\colon \mathbb{C} \to \mathscr{L}(X^n)$ with images in invertible operators such that

$$F(z) \circ (zI - \mathscr{P}(W)) \circ E(z) = \begin{bmatrix} W(z) & 0 & \cdots & \cdots & 0 \\ 0 & I & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & I \end{bmatrix}.$$

As a direct corollary of the Global Equivalence we get:

COROLLARY 3.2. *For every* $W(z) = Iz^n - \sum_{i=0}^{n-1} A_i z^i$, *where* $A_i \in \mathscr{L}(X)$, *we have*

$$\sigma(\mathscr{P}(W)) = \varsigma(W).$$

*Difference and Differential Equations*

In this subsection we are going to apply the operator $\mathscr{P}$ introduced previously in studying strong oscillation of solutions of difference and differential equations. The following corollary is a generalization of some results from [5] concerning oscillation of solution of linear difference equations in Banach spaces. We would like to mention that our method of proof is different from that applied in [5].

THEOREM 3.2. *Let* $W(z) = Iz^n - \sum_{i=0}^{n-1} A_i z^i$, *where* $A_i \in \mathscr{L}(X)$. *We assume that* $\varsigma(W) \cap \mathbb{R}_+ = \varnothing$.

*Then every solution of the difference equation*

$$x_{n+k} = \sum_{i=0}^{n-1} A_i x_{i+k} \qquad \textit{for} \quad k \in \mathbb{Z}_+ \tag{11}$$

*strongly oscillates.*

*Proof.* Let $\{x_k\}_{k \in \mathbb{Z}_+}$ be a given solution to the difference equation (11) and let $\xi \in X^*$ be arbitrary. To show that this solution strongly oscillates we have to show that $\{\xi(x_k)\}_{k \in \mathbb{Z}_+}$ strongly oscillates.

For $k \in \mathbb{Z}_+$ we put

$$v_k := (x_k, ..., x_{k+n-1})^T \in X^n,$$

where the upper index $T$ denotes transposition. One can verify directly by induction that

$$v_k = (\mathscr{P}(W))^k \, v_0 \qquad \text{for} \quad k \in \mathbb{Z}_+.$$

As $\sigma(\mathscr{P}(W)) = \varsigma(W)$ we obtain that that every point of $X^n$ strongly oscillates under $\mathscr{P}(W)$. This implies in particular that $v_0$ strongly oscillates. We now define $\tilde{\xi} \in (X^n)^*$ by

$$\tilde{\xi}((y_0, ..., y_{n-1})^T) = \xi(y_0) \qquad \text{for} \quad y_0, ..., y_{n-1} \in X.$$

As $v_0$ strongly oscillates we obtain that the sequence $\{\tilde{\xi}(\mathscr{P}(W)^k\, v_0)\}_{k \in \mathbb{Z}_+}$ strongly oscillates. But

$$\tilde{\xi}(\mathscr{P}(W)^k\, v_0) = \tilde{\xi}((x_k, \ldots, x_{k+n-1})^T) = \xi(x_k),$$

which yields that the sequence $\{\xi(x_k)\}_{k \in \mathbb{Z}_+}$ strongly oscillates. ∎

PROPOSITION 3.2. *Let* $W(z) = z^n - \sum_{i=0}^{n-1} A_i z^i$, *where* $A_i \in \mathscr{L}(X)$. *We assume that*

$$\varsigma_p(W) \cap \mathbb{R}_+ \neq \varnothing. \tag{12}$$

*Then there exists a solution of the difference equation*

$$x_{n+k} = \sum_{i=0}^{n-1} A_i x_{i+k} \qquad \text{for} \quad k \in \mathbb{Z}_+ \tag{13}$$

*which does not strongly oscillate.*

*Proof.* By the definition of $\varsigma_p$ and (12) there exists $\alpha \in \mathbb{R}_+$ and $x \in X$ such that

$$W(\alpha)(x) = \alpha x.$$

We define $x_k = \alpha^k x$. Then sequence $\{x_k\}_{k \in \mathbb{Z}_+}$ is a solution of (13) which does not oscillate strongly. ∎

THEOREM 3.3. *Let* $W(z) = Iz^n - \sum_{i=0}^{n-1} A_i z^i$, *where* $A_i \in \mathscr{L}(X)$. *We assume that* $\varsigma(W) \cap \mathbb{R} = \varnothing$.

*Then every solution of the differential equation*

$$d^n x/dt^n = \sum_{i=0}^{n-1} A_i\, d^i x/dt^i \tag{14}$$

*strongly oscillates.*

*Proof.* Let $x\colon \mathbb{R}_+ \to X$ be a given solution to the differential equation (14) and let $\xi \in X^*$ be arbitrary. To show that this solution strongly oscillates we have to show that $\xi \circ x : \mathbb{R}_+ \to \mathbb{R}$ strongly oscillates.

The function $v\colon \mathbb{R}_+ \to X^n$ defined by

$$v = (x, dx/dt, \ldots, d^{n-1}/dt^{n-1})^T$$

is a solution to

$$dv/dt = \mathcal{P}(W)\, v.$$

As $\sigma(\mathcal{P}(W)) = \varsigma(W)$, we obtain that $\sigma(\mathcal{P}(W)) \cap \mathbb{R} = \varnothing$, and the solution $v$ strongly oscillates. We define $\tilde{\xi} \in (X^n)^*$ by

$$\tilde{\xi}((y_0, \ldots, y_{n-1})^T) = \xi(y_0) \qquad \text{for} \quad y_0, \ldots, y_{n-1} \in X.$$

Then

$$\tilde{\xi}(v(s)) = \tilde{\xi}((x(s), dx/dt(s), \ldots, d^{n-1}x/dt^{n-1}(s))^T) = \xi(x(s)),$$

which yields that the function $\xi \circ x$ strongly oscillates. ∎

Proof of the following proposition is analogous to that of Proposition 3.2.

PROPOSITION 3.3. *Let* $W(z) = Iz^n - \sum_{i=0}^{n-1} A_i z^i$, *where* $A_i \in \mathcal{L}(X)$. *We assume that* $\varsigma_p(W) \cap \mathbb{R} = \varnothing$.

*Then there exists a solution of the differential equation*

$$d^n x/dt^n = \sum_{i=0}^{n-1} A_i\, d^i x/dt^i \tag{15}$$

*which does not strongly oscillate.*

## 4. STRONGLY CONTINUOUS SEMIGROUPS

### *Oscillation Is Generic*

As we have seen in the previous section, if the generator of the semigroup is bounded and its spectrum is disjoint with $\mathbb{R}_+$ then all points strongly oscillate.

The situation in the case when the generator is unbounded becomes much more complicated. We are first going to show that even if the generator has empty spectrum not all points have to strongly oscillate.

EXAMPLE 4.1. By $C(\mathbb{R}_+, \mathbb{R})$ we denote the space of all continuous functions from $\mathbb{R}_+$ to $\mathbb{R}$. Let $X = \{f \in C(\mathbb{R}_+, \mathbb{R}) : \lim_{r\to\infty} e^{r^2} f(r) = 0\}$ with the norm

$$\|f\| = \sup_{r \in \mathbb{R}_+} e^{r^2}\, |f(r)|.$$

One can easily notice that $X$ is a Banach space. By $T$ we denote the usual translation semigroup, that is

$$T(t)\, f(r) = T(t+r) \qquad \text{for} \quad r \in \mathbb{R}_+.$$

We will now show that $T$ is a strongly continuous semigroup. Let $f \in X$ and $\varepsilon > 0$ be arbitrary. Then by the definition of the space $X$ there exists $K > 0$ such that

$$|e^{r^2} f(r)| \leqslant \varepsilon \qquad \text{for} \quad r \geqslant K.$$

As the interval $[0, K]$ is compact $f$ is uniformly continuous on $[0, K]$, which implies that there exists $\delta > 0$ such that

$$e^{K^2}\, |f(r) - f(r+h)| \leqslant \varepsilon \qquad \text{for every} \quad r \in [0, K], h \in [0, \delta].$$

Then for every $h \in [0, \delta]$

$$\begin{aligned} \|T(h)\, f - f\| &= \sup\{|e^{r^2} f(r) - e^{r^2} f(r+h)| \mid r \in \mathbb{R}_+\} \\ &\leqslant 2 \sup\{|e^{r^2} f(r)| \mid r \in [K, \infty]\} \\ &\quad + e^{K^2} \sup\{|f(r) - f(r+h)| \mid r \in [0, K]\} \leqslant 3\varepsilon. \end{aligned}$$

As $\varepsilon$ was arbitrarily chosen this yields that $T$ is a strongly continuous semigroup.

Now we will show that

$$\|T(t)\| \leqslant e^{-t^2}. \tag{16}$$

So let $f \in X$ be arbitrary. Then

$$\begin{aligned} \|T(t)\, f\| &= \sup_{r \in \mathbb{R}_+} |e^{r^2} f(r+t)| \leqslant \sup_{r \in \mathbb{R}_+} |e^{r^2 - (r+t)^2} e^{(r+t)^2} f(r+t)| \\ &\leqslant \sup_{r \in \mathbb{R}_+} e^{-2tr - t^2} \cdot \sup_{r \in \mathbb{R}_+} |e^{(r+t)^2} f(r+t)| \\ &\leqslant e^{-t^2} \sup_{s \in [r, \infty)} |e^{s^2} f(s)| \leqslant e^{-t^2}\, \|f\|. \end{aligned}$$

This implies that $\|T(t)\| \leqslant e^{-t^2}$.

Due to (16) we have

$$\sqrt[n]{\|T(1)^n\|} = \sqrt[n]{\|T(n)\|} \leqslant e^{-n}$$

and therefore $r(T(1)) = \{0\}$ where $r$ denotes the spectral radius. This means that $\sigma(T(1)) = \{0\}$. As $e^{\sigma(A)} \subset \sigma(T(1))$, this yields that $\sigma(A) = \varnothing$. In consequence we obtain that $\sigma(A) \cap \mathbb{R} = \varnothing$.

Let $\xi \in X^*$ be defined by $\xi(f) = f(0)$ for $f \in X$ and let $g \in X$ be arbitrary such that $g(s) > 0$ for all $s \in \mathbb{R}_+$. Then $\xi(T(t)\, g) > 0$ for every $t \in \mathbb{R}_+$, which implies that $g$ does not strongly oscillate.

In view of the above example it seems natural to ask how "big" is the set of strongly oscillating points. We show that it is a residual set.

THEOREM 4.1. *Let $T$ be a strongly continuous semigroup and let $A$ denote its generator. We assume that $\sigma(A) \cap \mathbb{R} = \emptyset$. Then there exists a residual subset $S$ of $X$ such that every point of $S$ strongly oscillates.*

*Proof.* Let

$$S_n := \left\{ x \in X \mid d(-x, wedge\{orb_T(x)\}) < \frac{1}{n} \right\}$$

and let $S := \{x \in X \mid -x \in wedge\{orb_T(x)\}\}$. By Theorem 2.1 we know that $S$ denotes the set of all elements in $X$ which strongly oscillate. As $S = \bigcap_{n \in \mathbb{Z}_+} S_n$, to show that $S$ is residual it is enough to prove that every $S_n$ is a dense open set.

We first check that $S_n$ is open. So let $x \in S_n$ be arbitrary. By the definition of $S_n$ there exists $\varepsilon > 0$ such that

$$d(-x, wedge\{orb_T(x)\}) < \frac{1}{n} - \varepsilon.$$

This implies that there exist $\alpha_1, ..., \alpha_n \in \mathbb{R}_+$ and $t_1, ..., t_n \in \mathbb{R}_+$ such that

$$\left\| (-x) - \sum_{i=1}^{n} \alpha_i T(t_i)\, x \right\| < \frac{1}{n} - \frac{1}{2}\,\varepsilon.$$

Let $K := 1 + \sum_{i=1}^{n} |\alpha_i|\, \|T(t_i)\|$ and let $y \in B(x, \frac{\varepsilon}{2K})$ be arbitrary, where $B(a, r)$ denotes the closed ball with the center at $a$ and radius $r$. We have

$$\begin{aligned} d(-y, wedge\{orb_T(y)\}) &\leqslant \left\| (-y) - \sum_{i=1}^{n} \alpha_i T(t_i)\, y \right\| \\ &\leqslant \left\| (-x) - \sum_{i=1}^{n} \alpha_i T(t_i)\, x \right\| + \|x - y\| \\ &\quad + \sum_{i=1}^{n} |\alpha_i|\, \|T(t_i)\|\, \|x - y\| \\ &< \frac{1}{n} - \frac{\varepsilon}{2} + K\, \|x - y\| \leqslant \frac{1}{n} - \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \frac{1}{n}, \end{aligned}$$

which implies that $y \in S_n$.

Now we show that $S_n$ is dense. Let $x \in E$ and $\varepsilon > 0$ be arbitrary. Then there exists $x_\varepsilon \in dom(A)$ such that $\|x - x_\varepsilon\| < \frac{1}{2}\varepsilon$. Let $h > 0$ be chosen so that $h\,\|A(x_\varepsilon)\| < \min\{\frac{1}{n}, \frac{1}{2}\varepsilon\}$ and let

$$y = x_\varepsilon + hA(x_\varepsilon).$$

Then

$$\|x - y\| \leqslant \varepsilon.$$

We show that $y \in S_n$. By Proposition 2.1 we obtain that

$$-x_\varepsilon = -(I + hA)^{-1}\,(y) \in wedge(orb_T(y)).$$

By the definition of $y$ and the fact that $h\,\|A(x_\varepsilon)\| < \frac{1}{n}$ we obtain that $\|y - x_\varepsilon\| < \frac{1}{n}$, and consequently that $d(-y, wedge\{orb_T(y)\}) < \frac{1}{n}$. But this yields that $y \in S_n$, and consequently that $S_n$ is a dense subset of $X$.

As an intersection of the countable family of open dense sets we obtain that $S$ is a residual subset of $X$. ∎

*Global Oscillation for Groups*

Results of the previous subsection imply that although strong oscillation is typical, there may exist points which do not strongly oscillate. Thus there appears a natural question if we can impose some possibly weak assumptions on the semigroup to obtain strong oscillation of all points. We show that if the strongly continuous semigroup can be embedded into a strongly continuous group then all points strongly oscillate if the generator has no real eigenvalues.

LEMMA 4.1. *Let $T$ be a strongly continuous semigroup which generator $A$ satisfies*

$$\sigma(A) \cap \mathbb{R} = \varnothing.$$

*Let $x \in X$ be such that*

$$\lim_{\alpha \to \infty} \alpha(\alpha I + A)^{-1}\,x = x. \tag{17}$$

*Then $x$ strongly oscillates.*

*Proof.* The assertion of the Lemma is a direct consequence of Theorem 2.1 and Proposition 2.1. ∎

THEOREM 4.2. *Let $T$ be a strongly continuous group and let the generator $A$ of $T$ satisfy $\sigma(A) \cap \mathbb{R} = \emptyset$. Then every point of $X$ strongly oscillates.*

*Proof.* We are going to apply the previous lemma. To do so, let us notice that $-A$ is the generator of the strongly continuous semigroup $W$, where $W(t) = T(-t)$ for $t \in \mathbb{R}_+$. This implies that for every $x \in X$

$$\lim_{\alpha \to \infty} \alpha(\alpha I + A)^{-1} x = x. \tag{18}$$

Lemma 4.1 makes the proof complete. ∎

Now we would like to state a partial inverse of the above result (in particular for finite dimensional systems).

PROPOSITION 4.1. *Let $X$ be a strongly continuous semigroup. We assume that $\sigma_p(A) \cap \mathbb{R} \neq \emptyset$. Then there exists $x \in X$ which does not strongly oscillate.*

*Proof.* Let $\alpha \in \sigma_p(A) \cap \mathbb{R}$ be chosen arbitrarily. Then there exists $x \in X \setminus \{0\}$ such that $A(x) = \alpha x$. This means that $T(t)\, x = e^{\alpha t} x$, which yields that $orb_T(x) \subset \mathbb{R}_+ x$, and consequently that $-x \notin wedge\{orb_T(x)\}$. By Theorem 2.1 we obtain that $x$ does not strongly oscillate. ∎

We are going to apply the previous theorem in showing strong oscillation of the solutions of the wave equation on the bounded domain and in investigating when all points in a strongly continuous semigroup generated by a difference equation.

EXAMPLE 4.2. Suppose that $A$ is a selfadjoint operator on a Hilbert space $X$ such that $0 \in \rho(A)$. Then $iA$ generates a strongly continuous unitary group $T$ on $X$. Moreover, since $A$ is selfadjoint, $\sigma(iA) \subset i\mathbb{R}$. As $A$ is invertible, this yields that $\sigma(A) \cap \mathbb{R} = \emptyset$. By Theorem 4.2 we obtain that every point in $X$ strongly oscillates.

EXAMPLE 4.3. For more information concerning generation of strongly continuous groups by second order partial differential equations and the properties of Laplace operators we refer the reader to [3], [4], [11]. The following facts concerning the generation of strongly continuous group by the wave equation are well-known.

Let $\Omega$ be a bounded open domain in $\mathbb{R}^n$ with boundary $\Gamma$ of class $C^2$. Let us consider the wave equation, that is the following second-order linear hyperbolic problem in $w(t, x)$

$$\frac{\partial^2 w}{\partial t^2} = \Delta w$$

with boundary conditions $w(0, x) = w_0(x) \in H_0^1(\Omega)$, $w_t(0, x) = w_1(x) \in L^2(\Omega)$.

Let $A$: $L^2(\Omega) \supset \mathscr{D}(A) \to L^2(\Omega)$ be the operator defined by

$$Af = \Delta f, \mathscr{D}(A) = H_0^1(\Omega) \cap H^2(\Omega).$$

Then $A$ is self-adjoint and has compact resolvent $R(\cdot, A)$ such that $A^{-1} \in \mathscr{L}(L^2(\Omega))$. By the standard procedure we can now change the equation $\partial^2 w/\partial t^2 = Aw$ into a first order system

$$\frac{d}{dt}\begin{bmatrix} u \\ v \end{bmatrix} = \mathscr{A}\begin{bmatrix} u \\ v \end{bmatrix} \text{ on the space } W = H_0^1(\Omega) \times L^2(\Omega),$$

$$\mathscr{A} = \begin{bmatrix} 0 & I \\ A & 0 \end{bmatrix}, \mathscr{D}\mathscr{A} = (H_0^1(\Omega) \cap H^2(\Omega)) \times H_0^1(\Omega).$$

The space $W = H_0^1(\Omega) \times L^2(\Omega)$ is topologized with the usual Hilbert norm given explicitly by the equality

$$\|(u, v)\|_W^2 = \int_\Omega |\nabla u|^2 \, d\Omega + \int_\Omega |v|^2 \, d\Omega.$$

The norm $\| \; \|_W$ is sometimes called the energetic norm since it corresponds to the energy of the wave.

As is well-known $\mathscr{A}$ generates a strongly continuous unitary group on $W$. Moreover, the spectrum of $\mathscr{A}$ lies inn the imaginary axis and $0 \in \sigma(\mathscr{A})$ since $A^{-1} \in \mathscr{L}(L^2(\Omega))$. This implies that $\sigma(\mathscr{A}) \cap \mathbb{R} = \varnothing$.

Applying Theorem 4.2 we now obtain that whole space $H_0^1(\Omega) \times L^2(\Omega)$ strongly oscillates in the strongly continuous semigroup generated by $\mathscr{A}$.

We would like to mention that the following example can be generalized to the case when $A$ is a selfadjoint operator such that $-A$ is nonnegative and $0 \in \rho(A)$.

EXAMPLE 4.4. For the following results on the generation of strongly continuous semigroups by solutions of difference equations we refer the reader to [9].

Let $X$ be a finite dimensional Banach space and let the difference equation

$$x(t) = \sum_{k=0}^{n-1} A_k x(t - h_k) \tag{19}$$

be given, where $h_0 > h_1 > \cdots > h_{n-1} > 0$, $A_k \in \mathscr{L}(X)$ for $k = 0, \ldots, n-1$. We put $h = h_0$. By $\mathscr{C}$ we denote $\mathscr{C}([-h, 0], X)$—the Banach space of all continuous functions from interval $[-h, 0]$ to $X$ with supremum norm. We define

$$\mathscr{C}_D = \left\{ x \in \mathscr{C} : x(0) = \sum_{k=0}^{n-1} A_k x(-h_k) \right\}.$$

Then for each function $x \in C_D$ there corresponds a unique solution $\bar{x}\colon [-h, \infty) \to X$ of (19) such that $\bar{x}|_{[-h,0]} = x$. For $t \in \mathbb{R}_+$ we define the function $x_t \in \mathscr{C}_D$ by the formula $x_t(s) = \bar{x}(t+s)$ for $s \in [-h, 0]$. Then it can be easily verified that the function $T\colon \mathbb{R}_+ \times \mathscr{C}_D \ni x \to x_t \in \mathscr{C}_D$ is a strongly continuous semigroup. Moreover, the spectrum of the generator $A$ of semigroup $T$ is given by the zeros of the characteristic function of (19):

$$\sigma(A) = \left\{ z \in \mathbb{C} : I - \sum_{k=0}^{n-1} e^{-h_k z} A_k \text{ is not invertible} \right\}.$$

So let us assume that $\sigma(A) \cap \mathbb{R} = \varnothing$. Then by Theorem 4.1 we know that almost all points in $\mathscr{C}_D$ strongly oscillate. Applying Theorem 4.2 we will show that all points in $\mathscr{C}_D$ strongly oscillate iff $A_0$ is invertible.

First let us suppose that $A_0$ is invertible. Then $T$ can be extended to a strongly continuous group—the equation (19) can be written in the equivalent form

$$x(t) = A_0^{-1} x(t+h_0) + \sum_{k=1}^{n-1} A_0^{-1} A_k x(t + (h_0 - h_k)),$$

from which one easily sees that all solutions of (19) can be uniquely extended backwards. For $t$ negative we now define values of $T(t)\,x$ analogously as in the definition for positive $t$. Theorem 4.2 yields that all points in $\mathscr{C}_D$ strongly oscillate.

So let us now consider the case when $A_0$ is not invertible. Let $a \in ker(A_0)$ be arbitrary and let $f \in X^*$ be chosen so that $f(a) = 1$. For every $r \in \mathbb{R}$ we define the function $x_r \in \mathscr{C}_D$ by the formula

$$x_r(s) := \max\{r - s, 0\}\, a \qquad \text{for} \quad s \in [-h, 0].$$

Then one can easily verify that $T(t)\, x_{-h_1} = x_{-h_1 - t}$.

Let us now define $f_D \in (\mathscr{C}_D)^*$ by

$$f_D(x) = f(x(-h)) \qquad \text{for} \quad x \in \mathscr{C}_D.$$

Then

$$f_D(T(t)\, x_{-h_{k-2}}) = \max\{h - h_1 - t, 0\}.$$

However, this clearly implies that the point $x_{-h_1}$ does not strongly oscillate.

*Properties of Nonoscillating Points*

We have seen in Example 4.1 that even if the generator of a strongly continuous semigroup has no real eigenvalues then not all points in the space have to strongly oscillate. Thus it seems reasonable to study the behaviour of such points. We show that points which do not strongly oscillate have to be in some sense similar to that constructed in Example 4.1, namely they have to decrease to zero faster then every exponential function.

The following definition is a modification of the definition of small solutions from [8] (see also [2]). According to [8] the function is small if it goes faster to zero then every exponential function. We need functions which in integral sense decrease to zero faster then every exponential.

DEFINITION 4.1. Let $f: \mathbb{R}_+ \to \mathbb{R}$. We say that $f$ is *integrally small* if

$$\lim_{r \to \infty} \int_0^r f(t)\, e^{kt}\, dt \text{ exists for every } k \in \mathbb{R}.$$

Let $\xi \in X^*$ be arbitrary and let $f: \mathbb{R}_+ \to X$. We say that $f$ is $\xi$-integrally small if $\xi \circ f$ is integrally small. Suppose that we are given a semidynamical system $T$ with continuous time in $X$. Then we say that $x \in X$ is *$\xi$-integrally small* if $T(\cdot)\, x$ is $\xi$-integrally small.

Now we will state main result in this subsection. It says that if a point does not strongly oscillate then there exists a functional $\xi$ such that $x$ is a nonzero $\xi$-integrally small solution. As the proof of this theorem is not direct we will divide it into a few lemmas and provide it at the end of this subsection.

THEOREM 4.3. *Let $T$ be a strongly continuous semigroup and let $A$ be the generator of $T$. We assume that*

$$\sigma(A) \cap \mathbb{R} = \varnothing.$$

*Let $x \in X$ and $\xi \in X^*$ be such that $\xi(T(\cdot)\, x)$ is eventually nonnegative. Then $x$ is $\xi$-integrally small.*

As an immediate corollary we have the following important property of points which do not strongly oscillate.

COROLLARY 4.1. *Let $T$ be a strongly continuous semigroup and let $A$ be the generator of $T$. We assume that*

$$\sigma(A) \cap \mathbb{R} = \varnothing.$$

*Let $x$ be a point in $X$ which does not strongly oscillate. Then there exists $\xi \in X^*$ such that $\xi(T(\cdot)\, x)$ is nonzero and $x$ is $\xi$-integrally small.*

We now need to quote some results and definitions from the theory of Laplace Transform (see [13]).

Let $f: \mathbb{R}_+ \to \mathbb{C}$ be a locally integrable function. By $\Omega_f \subset \mathbb{C}$ we denote the set of all $\omega \in \mathbb{C}$ such that

$$\mathscr{L}(f)(\omega) := \lim_{r \to \infty} \int_0^r e^{-ws} f(s)\, ds$$

exists. The function $\mathscr{L}(f): \Omega_f \to \mathbb{C}$ is called a *Laplace transform* of $f$. $\Omega_f$ is a half plane, that is there exists $\sigma_c(f) \in \mathbb{R} \cup \{-\infty\} \cup \{\infty\}$, called the *abscissa of convergence of $f$*, such that $z \in \Omega_f$ for every $z \in \mathbb{C}$ with $re(z) > \sigma_c(f)$, and $z \notin \Omega_f$ for every $z \in \mathbb{C}$ with $re(z) < \sigma_c(f)$.

Thus a function $f: \mathbb{R}_+ \to \mathbb{C}$ is *integrally small* iff $\sigma_c(f) = -\infty$. This yields in particular that the Laplace transform of an integrally small function is an entire function.

The following property of Laplace transform will be necessary in the characterization of non-oscillating points.

Let $\Omega$ be an open convex subset of $\mathbb{C}$. We say that a holomorphic function $f: \Omega \to \mathbb{C}$ has a singularity at a given point $z \in \partial\Omega$ if $f$ does not have a holomorphic continuation to a neighborhood of $z$.

THEOREM La (Theorem 10.1, Sect. 5 from [13]). *Let $f$ be a nonnegative function. Suppose that $\sigma_c(f) \in \mathbb{R}$. Then the Laplace transform has a singularity in $\sigma_c(f)$.*

This implies that if a function $f$ is eventually positive and $\sigma_c(f) \in \mathbb{R}$ then the Laplace transform of $f$ has a singularity at $\sigma_c(f)$.

Now we are ready to prove Theorem 4.3.

*Proof* (of Theorem 4.3). Let $z \in \mathbb{C}$, $re(z) > type(T)$ be arbitrary. Then $\lim_{r \to \infty} \int_0^r e^{-zs} T_{\mathbb{C}}(s)\, x\, ds$ exists and

$$(zI - A_{\mathbb{C}})^{-1}(x) = \int_0^\infty e^{-zs} T_{\mathbb{C}}(s)\, x\, ds,$$

where the subscript $\mathbb{C}$ denotes complexification. This implies that

$$\xi_{\mathbb{C}}((zI_{X_{\mathbb{C}}} - A_{\mathbb{C}})^{-1}(x)) = \int_0^\infty e^{-zs}\xi_{\mathbb{C}}(T_{\mathbb{C}}(s)\,x)\,ds. \tag{20}$$

Let $r_0 = \sigma_c(\xi_{\mathbb{C}}(T_{\mathbb{C}}(\cdot)\,x)) = \sigma_c(\xi(T(\cdot)\,x) \in \mathbb{R} \cup \{-\infty\} \cup \{\infty\}$. Clearly $r_0 \leqslant$ $type(T)$. We show that $r_0 = -\infty$.

For an indirect proof let us assume that this is not the case. By the assumptions we know that there exists $p \in \mathbb{R}_+$ such that $\xi(T(s)\,x) \geqslant 0$ for $s \geqslant p$. Let

$$f(s) := \begin{cases} 0 & \text{for} \quad s < p \\ \xi(T(s)\,x) & \text{for} \quad s \geqslant p. \end{cases}$$

Then $\sigma_c(f) = r_0$. As $f$ is a nonnegative function by Theorem La we obtain that $\mathscr{L}(f)$ has a singularity at $r_0$, which clearly yields that $\mathscr{L}(\xi(T(\cdot)\,x))$ has a singularity at $r_0$.

Let $r_1 > type(T)$ be arbitrary. As $\sigma(A) \cap \mathbb{R} = \varnothing$ there exists an open convex set $O \subset \mathbb{C}$ containing $(r_0, r_1)$ and such that $re(O) > r_0$, $\sigma(A) \cap O = \varnothing$. By (20) we obtain that $\mathscr{L}(\xi_{\mathbb{C}}(T_{\mathbb{C}}(\cdot)\,x)) = \xi_{\mathbb{C}}(zI_{X_{\mathbb{C}}} - A_{\mathbb{C}})^{-1}(x)$ for $z \in O$, $re(z) \geqslant type(T)$. As $O$ is open convex and both the above functions are defined on $O$ and are equal in an open subset of $O$ they coincide. This yields contradiction as $\mathscr{L}(\xi_{\mathbb{C}}(T_{\mathbb{C}}(\cdot)\,x))$ has a singularity at $r_0$ and $\xi_{\mathbb{C}}(zI_{X_{\mathbb{C}}} - A_{\mathbb{C}})^{-1}(x)$ is holomorphic at neighborhood of the point $r_0$.

Thus we have showed that $\sigma_c(f) = -\infty$, which implies that $x$ is a $\xi$-integrally small solution. ∎

*Retarded Functional-Differential Equations*

We are going to present the generalization of some results from [1] and [5] concerning oscillation of solutions of delay differential equations. For the convenience of the reader and to establish notation we quote some basic notions and properties of delay difference equations. For more information concerning this broad subject we refer the reader to [2], [6], [7].

In this subsection we assume that $X$ is a finite dimensional Banach space. By $NBV([0,h], \mathscr{L}(X))$ we denote the set of all normalized bounded variaton functions on interval $[0,h]$ with values in $\mathscr{L}(X)$. We shall study

$$\dot{x}(t) = \int_0^h d\zeta(\theta)\,x(t-\theta), \tag{21}$$

where $\zeta \in NBV([0, h], X)$ is fixed. As an initial condition we take $\phi \in \mathscr{C} = C([-h, 0], X)$ and assume that

$$x(\theta) = \phi(\theta), \qquad -h \leqslant \theta \leqslant 0. \tag{22}$$

A *solution* to the initial value problem (21)–(22) on the interval $I = [-h, \infty)$, is a function $x \in C(I, X)$ such that

(i) (22) holds;
(ii) $x$ is continuously differentiable on $(0, \infty)$ and (21) holds;
(iii) $\lim_{t \downarrow 0} (x(t) - \phi(0))$ exists and equals $\int_0^h d\zeta(\theta)\, \phi(-\theta)$.

For a given RFDE we define its *characteristic function* $\Delta(z)\colon \mathbb{C} \to \mathscr{L}(X)$ by

$$\Delta(z) = zI - \int_0^h e^{-zt}\, d\zeta(t).$$

$\Delta$ is an entire function.

Now we are ready to quote Lemma 1 from [1].

THEOREM Ar. *Consider a linear scalar functional differential equation*

$$\dot{x}(t) = \int_0^h d\zeta(\theta)\, x(t-\theta).$$

*Then each solution $x$ of the above scalar equation oscillates iff the characteristic equation has no real roots.*

Similar type of results can be obtained also in the vector case (see [5]).

As is well-known each linear RFDE generates a strongly continuous semigroup $T$ in the space $\mathscr{C}$. Thus it seems reasonable to investigate strong oscillation for points in $\mathscr{C}$. We are going to generalize Theorem Ar in this direction.

As usual in the theory of delay equations we write

$$x_t(\theta) := x(t+\theta) \qquad \text{for} \quad t \geqslant 0 \qquad \text{and} \qquad -h \leqslant \theta \leqslant 0.$$

With this notation $x_t \in \mathscr{C}$ is the state at time $t$. For each initial condition $\phi \in \mathscr{C}$ and $t \geqslant 0$ we define

$$T(t)\, \phi := x_t.$$

It is well-known that then $T$ is a strongly continuous semigroup in $\mathscr{C}$. The spectrum of the generator $A$ of the semigroup $T$ satisfies the following important formula

$$\sigma(A) = \{z \in \mathbb{C} \mid \Delta(z) \text{ is not invertible in } \mathscr{L}(X)\}.$$

From the previous results we know that if the spectrum of the generator of a strongly continuous semigroup is disjoint with $\mathbb{R}$ then strong oscillation is generic.

Now we will show an example that in the case of strongly continuous semigroups generated by functional differential equation not all points strongly oscillate.

EXAMPLE 4.5. Let $\mathscr{C} = C([-2, 0], \mathbb{R}^2)$. We consider the semigroup in $\mathscr{C}_{\mathbb{R}}$ generated by the equation

$$\begin{bmatrix} \dot{u}(t) \\ \dot{v}(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} u(t-2) \\ v(t-2) \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u(t-1) \\ v(t-1) \end{bmatrix}. \tag{23}$$

One can easily notice that the characteristic function of this equation has no real eigenvalues.

For $(u, v) \in \mathscr{C}$ we define $\xi \in \mathscr{C}^*$ by the formula

$$\xi(u, v) := u(-2).$$

Let $u\colon [-2, 0] \to \mathbb{R}$ be defined by

$$u(t) := \begin{cases} (t-1)^2 & \text{for } t \in [-2, -1] \\ 0 & \text{for } t \in [-1, 0]. \end{cases}$$

Let us now notice that $(u, 0) \in \mathscr{C}$ does not strongly oscillate in the semigroup $T$ generated by (23), as $\xi((u, 0)) = 1$ but $\xi(T(s)(u, 0)) \geqslant 0$ for every $s \in \mathbb{R}_+$.

Now we are going to obtain main result of our section, a generalization of [1]. We first need a generalization of the results of D. Henry from [8]. The following result follows directly from the proof of Lemma 4.2, Chapter V from [2].

THEOREM H. *Let $T$ be a strongly continuous semigroup generated by RFDE. Let $x \in \mathscr{C}$ and $\xi \in \mathscr{C}^*$ be arbitrary. If $\xi(T(\cdot)\,x)$ is integrally small then*

$$\xi(T(t)\,x) = 0 \qquad \textit{for} \quad t \geqslant h \cdot \mathit{dim}(X).$$

As a corollary we obtain the following result:

THEOREM 4.4. *Let $T$ be a strongly continuous semigroup with the generator $A$ generated by an RFDE in the space $\mathscr{C} = C([-h, 0], X)$, where $X$ is an n-dimensional Banach space. Then for every $x \in \mathscr{C}$ the point $T(nh)\,x$ strongly oscillates iff*

$$\sigma(A) \cap \mathbb{R} = \emptyset.$$

*Proof.* Let $x \in \mathscr{C}$ and $\xi \in \mathscr{C}^*$ be arbitrarily chosen. Suppose that the function $\mathbb{R}_+ \ni t \to \xi(T(t)\,x) \in \mathbb{R}$ does not strongly oscillate. This implies by Theorem 4.3 that $\xi(T(t)\,x)$ is an integrally small function, which by Theorem H yields that $\xi(T(t)\,x) = 0$ for $t \geqslant nh$.

We show the opposite implication. So suppose that

$$\sigma(A) \cap \mathbb{R} = \emptyset.$$

Since, as is well-known the point spectrum of $A$ coincides with spectrum, so there exists an $\alpha \in \mathbb{R}$ and an $x \in \mathscr{C} \setminus \{0\}$ such that

$$A(x) = \alpha x.$$

Then $T(t)\,x = e^{\alpha t}x$, which implies that $T(nh)\,x$ does not strongly oscillate in the semidynamical system $T$. ∎

## REFERENCES


1. O. Arino, Oscillation criteria in delay equations, *J. Differential Equations* **53** (1984), 115–123.
2. O. Diekmann, S. A. van Gils, S. M. Verduyn Lunel, and H.-O. Walther, "Delay Equations," Applied Mathematical Sciences, Vol. 110, Springer-Verlag, Berlin/New York.
3. H. O. Fattorini, "Second Order Linear Differential Equations in Banach Spaces," North-Holland Mathematics Studies, Vol. 108, North-Holland, Amsterdam.
4. I. Gohberg, S. Goldberg, and M. Kaashoek, "Classes of Linear Operators, Vol I," Operator Theory Advances and Application, Vol. 49, Birkhauser, Basel.
5. I. Györi and G. Ladas, "Oscillation Theory of Delay Differential Equations," Clarendon Press, Oxford, 1991.
6. J. Hale, "Theory of Functional Differential Equations," Springer-Verlag, New York, 1977.
7. J. Hale and V. Lunel, "Introduction to Theory of Functional Differential Equations," Applied Mathematical Sciences, Vol. 99, Springer-Verlag, New York, 1993.
8. D. Henry, Small solutions of linear autonomous functional differential equations, *J. Differential Equations* **8** (1970), 494–501.

9. D. Henry, Linear autonomous neutral functional differential equations, *J. Differential Equations* **15** (1974), 106–128.
10. V. Kozlov and V. Maz'ya, "Differential Equations with Operator Coefficients," Springer-Verlag, Berlin/New York, 1999.
11. A. Pazy, "Semigroups of Linear Operators and Applications to Partial Differential Equations," Applied Mathematical Sciences, Vol. 44, Springer-Verlag, Berlin/New York.
12. J. Tabor, Oscillations in Banach spaces and positive operators, *J. Differential Equations* **144** (1998), 219–232.
13. D. V. Widder, "An Introduction to Transform Theory," Academic Press, New York/London, 1971.